

Formalizing Bombieri Vinogradov

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Chapter 1

Bombieri–Vinogradov

1.1 Statement

The main result is the Bombieri–Vinogradov theorem, which gives a strong estimate for primes in arithmetic progressions on average over moduli.

Theorem 1. For each fixed $A \geq 0$,

$$\sum_{q \leq Q} \max_{y \leq x} \max_{a \in (\mathbb{Z}/q\mathbb{Z})^*} \left| \psi(y; q, a) - \frac{y}{\varphi(q)} \right| \ll_A \frac{x}{(\log x)^A}$$

uniformly for all $x \geq 2$ and $1 \leq Q \leq \frac{\sqrt{x}}{(\log x)^{A+3}}$.

Proof. Apply 7 and absorb the error terms using 9. □

1.2 Assumptions

We take Siegel–Walfisz and the large sieve inequality as axioms.

Assumption 2. Let $A, C > 0$. If $1 \leq q \leq (\log x)^C$ and $a \in (\mathbb{Z}/q\mathbb{Z})^*$, then

$$\psi(x; q, a) = \frac{x}{\varphi(q)} + O_{A,C} \left(\frac{x}{(\log x)^A} \right).$$

Assumption 3. Let $Q \geq 1$, $H \in \mathbb{Z}$, $N \in \mathbb{Z}_{\geq 1}$ and $c = (c_{H+1}, \dots, c_{H+N}) \in \mathbb{C}^N$. We then have

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}}^* \frac{q}{\varphi(q)} \left| \sum_{H < n \leq H+N} c_n \chi(n) \right|^2 \ll (N + Q^2) \|\vec{c}\|_2^2,$$

1.3 Definitions

Definition 4.

$$\Delta_f(x; q, a) := \sum_{n \leq x, n \equiv a \pmod{q}} f(n) - \frac{1}{\varphi(q)} \sum_{n \leq x, (n, q) = 1} f(n)$$

for $x \geq 1$, $q \in \mathbb{N}$

Theorem 5.

$$\Delta_f(y; q, a) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}, \chi \neq \chi_0} \bar{\chi}(a) \sum_{n \leq y} f(n) \chi(n)$$

Proof. □

Definition 6. For $f : \mathbb{N} \rightarrow \mathbb{R}$ and $r : \mathbb{N}$ we use f_r to denote $n \mapsto f(n)1_{(n,r)=1}$

1.4 Reduction to Δ_Λ

Bombieri–Vinogradov for ψ is reduced to a bound for Δ_Λ via the following steps.

Theorem 7. For each fixed $A \geq 0$ we have

$$\sum_{q \leq Q} \max_{\sqrt{x} \leq y \leq x} \max_{a \in (\mathbb{Z}/q\mathbb{Z})^*} |\Delta_\Lambda(y; q, a)| \ll_A \frac{x}{(\log x)^A}$$

uniformly for $x \geq 2$ and $1 \leq Q \leq \sqrt{x}/(\log(x))^{A+3}$

Proof. Follows from 13 and the triangle inequality, combining the bounds 16, 23, and 35. □

Theorem 8.

$$\Delta_\Lambda(x; q, a) = \psi(x; q, a) - \frac{1}{\varphi(q)} \sum_{n \leq x, n \nmid q} \Lambda(n)$$

Proof. □

Theorem 9. For all $A \in \mathbb{N}$,

$$\sum_{n \leq x, n \nmid q} \Lambda(n) = x + O_A(x/\log(x)^A + \log q \log x)$$

Proof. □

1.5 Vaughan’s Identity

The von Mangoldt function decomposes as $\Lambda = \Lambda^\sharp + \Lambda^\flat + \Lambda_{\leq U}$, where $UV \leq \sqrt{x}$ and $U, V \geq e^{\sqrt{\log x}}$.

Definition 10. $\Lambda^\sharp = \mu_{\leq V} * \log - (\Lambda_{\leq U} * \mu_{\leq V}) * 1$

Definition 11. $\Lambda^\flat = (\Lambda_{> U} * 1) * \mu_{> V}$

Definition 12. $\Lambda_{\leq U} = 1_{\leq U} \cdot \Lambda$

Theorem 13. Decompose $\Lambda = \Lambda^\sharp + \Lambda^\flat + \Lambda_{\leq U}$

Proof. □

1.6 Small Primes ($\Lambda_{\leq U}$)

The contribution of the small part $\Lambda_{\leq U}$ is bounded trivially.

Lemma 14.

$$\sum_{n \leq y} \Lambda(n) \ll U \log x$$

Proof.

□

Lemma 15. For $y, U > 0$, $q \in \mathbb{N}$ and $a \in \mathbb{Z}/q\mathbb{Z}$,

$$|\Delta_{\Lambda_{\leq U}}(y; q, a)| \ll U \log x$$

Proof.

□

Theorem 16. For each fixed $A \geq 0$, $x \geq 2$ and $1 \leq Q \leq \sqrt{x}/(\log x)^{A+3}$,

$$\sum_{q \leq Q} \max_{\sqrt{x} \leq y \leq x} \max_{a \in (\mathbb{Z}/q\mathbb{Z})^*} |\Delta_{\Lambda_{\leq U}}(y; q, a)| \leq Q \sqrt{x} \ll_A \frac{x}{(\log x)^{A+2}}$$

Proof.

□

1.7 Type I Sums ($\Lambda^\sharp$)

The sharp part $\Lambda^\sharp$ is a type I sum, bounded via Abel summation.

Lemma 17. If f is an arithmetic function supported on $[1, y]$ then

$$\Delta_{f * g}(x; q, a) = \sum_{\substack{k \leq y \\ (k, q) = 1}} f(k) \Delta_g\left(\frac{x}{k}; q, a\bar{k}\right)$$

Proof.

□

Lemma 18. For $x \geq 1$, $q \in \mathbb{N}$ and $a \in (\mathbb{Z}/q\mathbb{Z})^*$,

$$|\Delta_1(x; q, a)| \leq 1$$

Proof. Carefully consider length q intervals. Alternatively, write

$$\Delta_1(t; q, a) = \frac{1}{\varphi(q)} \sum_{a' \in (\mathbb{Z}/q\mathbb{Z})^*} \left(\sum_{\substack{n \leq t \\ n \equiv a \pmod{q}}} 1 - \sum_{\substack{n \leq t \\ n \equiv a' \pmod{q}}} 1 \right)$$

and note each inner difference is bounded by 1 in absolute value.

□

Lemma 19. If g is continuously differentiable on $[1, x]$ then

$$\Delta_g(x; q, a) = \Delta_1(x; q, a) g(x) - \int_1^x \Delta_1(t; q, a) g'(t) dt$$

Proof. By Abel summation.

□

Lemma 20. *If g is continuously differentiable and monotone on $[1, x]$ with $g(1) = 0$, then for all $t \geq 1$ and $a \in (\mathbb{Z}/q\mathbb{Z})^*$,*

$$|\Delta_g(x; q, a)| \leq 2g(x)$$

Proof. □

Theorem 21. *Let $v \geq 0$ and let f be an arithmetic function supported on $[1, x]$. For $x \geq 2$, $q \in \mathbb{N}$ and $a \in (\mathbb{Z}/q\mathbb{Z})^*$,*

$$|\Delta_{f * \log^v}(x; q, a)| \leq 2(\log x)^v \sum_{k \leq x} |f(k)|$$

Proof. Straightforward application of the previous lemmas. □

Theorem 22. *For $U, V \geq 1$, $x \geq 2$, $q \in \mathbb{N}$, $r \leq x$ and $a \in (\mathbb{Z}/q\mathbb{Z})^*$,*

$$\max_{y \leq x} \max_{a \in (\mathbb{Z}/q\mathbb{Z})^*} |\Delta_{\Lambda_r^\sharp}(y; q, a)| \ll UV \log x$$

Proof. Apply 21 twice: once with $f = \mu_{\leq V, r}$, $v = 1$, and once with $f = \Lambda_{\leq U} * \mu_{\leq V, r}$, $v = 0$, $y = UV$. □

Theorem 23. *For each fixed $A \geq 0$, $x \geq 2$ and $1 \leq Q \leq \sqrt{x}/(\log x)^{A+3}$,*

$$\sum_{q \leq Q} \max_{\sqrt{x} \leq y \leq x} \max_{a \in (\mathbb{Z}/q\mathbb{Z})^*} |\Delta_{\Lambda^\sharp}(y; q, a)| \ll_A \frac{x}{(\log x)^A}$$

Proof. □

1.8 Type II Sums (Λ^b)

1.8.1 Definitions

Definition 24.

$$S_r(y, \xi) := \left| \sum_{n \leq y} \Lambda_r^b(n) \xi(n) \right|$$

Definition 25.

$$T_r(x, Q) := \sum_{(\log x)^C < d \leq Q/r} \frac{1}{\varphi(d)} \sum_{\xi \pmod{d}}^* \max_{\sqrt{x} \leq y \leq x} S_r(y, \xi)$$

1.8.2 Reduction to character sums

Lemma 26. *This is a standard result. Let f be a function from Dirichlet characters. Then*

$$\sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}} f(\chi) = \sum_{d|q} \sum_{\xi \pmod{d}}^* f(1_{(n,q)=1} \xi)$$

Proof. □

Lemma 27. *Let f be an arithmetic function. For $r \leq x$, $q > 1$ and $(a, q) = 1$,*

$$\sum_{\xi \pmod{q}}^* \bar{\xi}(a) \sum_{n \leq y} \xi(n) f_r(n) = \sum_{d|q} \mu(q/d) \varphi(d) \Delta_{f_{rq}}(y; d, a)$$

Proof. Fix $P \in \mathbb{N}$ with $q \mid P$. Define F_P and G_P on divisors of P by

$$F_P(q) := \sum_{\substack{\chi \neq \chi_0 \\ (\bmod q)}} \bar{\chi}(a) \sum_{n \leq y} \chi(n) f_{rP}(n) = \Delta_{f_{rP}}(y; q, a),$$

$$G_P(d) := \sum_{\substack{\xi \\ (\bmod d)}}^* \bar{\xi}(a) \sum_{n \leq y} \xi(n) f_{rP}(n) \quad (d > 1), \quad G_P(1) = 0.$$

Since every non-principal character mod q factors through a unique primitive character, $F_P(q) = \sum_{d \mid q} G_P(d)$. By Möbius inversion,

$$G_P(q) = \sum_{d \mid q} \mu(q/d) F_P(d) = \sum_{d \mid q} \mu(q/d) \Delta_{f_{rP}}(y; d, a).$$

Set $P = q$ to conclude. □

Lemma 28.

$$|\Delta_{\Lambda^b}(y; q, a)| \leq \frac{1}{\varphi(q)} \left| \sum_{\substack{d \mid q \\ 1 < d \leq (\log x)^C}} \sum_{s \mid d} \mu(d/s) \varphi(s) \Delta_{\Lambda_q^b}(y; s, a) \right| + \frac{1}{\varphi(q)} \sum_{\substack{d \mid q \\ d > (\log x)^C}} \sum_{\xi \pmod{d}}^* S_{q/d}(y, \xi)$$

Proof. □

Theorem 29.

$$\frac{1}{\varphi(q)} \left| \sum_{\substack{d \mid q \\ 1 < d \leq (\log x)^C}} \sum_{s \mid d} \mu(d/s) \varphi(s) \Delta_{\Lambda_q^b}(y; s, a) \right| \ll_{A,C} \frac{x}{\varphi(q) (\log x)^{A+1}}$$

Proof. Push the absolute values inside, then

$$\begin{aligned} \sum_{d \mid q, d \leq (\log x)^C} \sum_{s \mid d} \varphi(s) |\Delta_{\Lambda_q^b}(y; s, a)| &\ll_{A,C} \sum_{d \leq (\log x)^C} \left(\sum_{s \mid d} \varphi(s) \right) \frac{x}{(\log x)^{A+2C+1}} \\ &\ll \frac{x}{(\log x)^{A+2C+1}} \sum_{d \leq (\log x)^C} d \\ &\ll \frac{x}{(\log x)^{A+1}}. \end{aligned}$$

□

Theorem 30.

$$\sum_{q \leq Q} \max_{\substack{\sqrt{x} \leq y \leq x \\ a \in (\mathbb{Z}/q\mathbb{Z})^*}} |\Delta_{\Lambda^b}(y; q, a)| \leq \sum_{r \leq Q} \frac{T_r(x, Q)}{\varphi(r)} + O\left(\frac{x}{(\log x)^A}\right)$$

Proof. Sum the error from 29 over $q \leq Q$ using $\sum_{n \leq x} 1/\varphi(n) \ll \log x$, then regroup the main sum by $r = q/d$. □

1.8.3 Large sieve estimates

Theorem 31. *Let f and g be arithmetic functions supported on $[1, M]$ and $[1, N]$ respectively. For $x, Q \geq 1$,*

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}}^* \frac{q}{\varphi(q)} \max_{y \leq x} \left| \sum_{n \leq y} (f * g)(n) \chi(n) \right| \ll (\sqrt{MN} + \sqrt{M}Q + \sqrt{N}Q + Q^2) (\log x) \|f\|_2 \|g\|_2$$

Proof. Uses Cauchy–Schwarz and the large sieve inequality (3). The proof in the book uses the classical version of Perron’s integral formula as $1_{n \leq x} = \int_{-T}^T \frac{(x/n)^{\alpha+it}}{\alpha+it} dt / (2\pi) + O(\dots)$ But we have a different version in PNT+. I haven’t worked out how this changes the proof yet. $\square$

Lemma 32.

$$\Lambda^b(n) = \sum_{U < 2^j \leq 2x/V} (f_j * g_j)(n) \quad \text{for } n \leq x,$$

where $f_j(k) = (\Lambda_{>U} * 1)(k) 1_{2^{j-1} < k \leq 2^j}$ and $g_j(\ell) = \mu(\ell) 1_{V < \ell \leq x/2^{j-1}}$.

Proof. $\square$

Theorem 33. *For $x, Q \geq 2$, $U, V \in [1, x]$ and $r \in \mathbb{N}$,*

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}}^* \frac{q}{\varphi(q)} \max_{y \leq x} S_r(y, \chi) \ll \left(x + \frac{Qx}{\sqrt{U}} + \frac{Qx}{\sqrt{V}} + Q^2 \sqrt{x} \right) (\log x)^3$$

Proof. Apply the dyadic decomposition 32 (restricted to integers coprime to r) and apply 31 to each dyadic piece. When summing over j note $U \leq 2^j$, so $\sum_{U \leq 2^j} 2^{-j/2} \ll 1/\sqrt{U}$. $\square$

Theorem 34.

$$T_r(x, Q) \ll \frac{x}{(\log x)^{C-3}} + \frac{x(\log x)^4}{\sqrt{U}} + \frac{x(\log x)^4}{\sqrt{V}} + \frac{Q\sqrt{x}(\log x)^3}{r}$$

Proof. Divide the sum defining T_r into dyadic intervals in d and apply 33. $\square$

Theorem 35. *For each fixed $A \geq 0$, $x \geq 2$ and $1 \leq Q \leq \sqrt{x}/(\log x)^{A+3}$,*

$$\sum_{q \leq Q} \max_{\sqrt{x} \leq y \leq x} \max_{a \in (\mathbb{Z}/q\mathbb{Z})^*} |\Delta_{\Lambda^b}(y; q, a)| \ll_A \frac{x}{(\log x)^A}$$

Proof. Plug the bound from 34 into 30, then choose $U = V = e^{\sqrt{\log x}}$ and $C = A + 4$. $\square$